On Maximum Drag in Supersonic Flow

YU. D. SHMYGLEVSKY

Computing Centre of the USSR Academy of Sciences, Moscow

The wave drag of a body in a steady supersonic gas flow equals zero if the body does not initiate the shock waves and the flow is non-separable. Busemann's biplane is an example. The simple investigation, when the detailed structure of the flow is not taken into account, allows the upper limit of the wave drag at the given sizes of a body to be determined.

According to the momentum law we have the relation between the power action of a gas flow on a body and the deviation angle of the jet from the initial direction. To obtain the maximum drag it is necessary to find the best manner of the flow reverse by using the maximum mass flow. The solution of the problem can be based on the flow scheme on Fig. 1. The gas captured

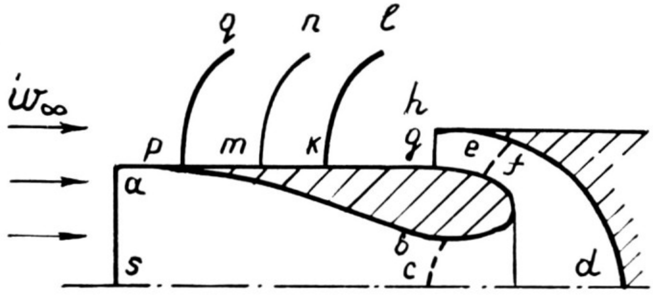

FIG. 1.

by the diffuser is thrown away towards the initial flow. The lines *kl* and *pq* are shock waves and the line *mn* is a dividing stream line. The interaction of the two parts of the flow in the region *qpkl* is complicated. At the beginning of the investigation it is expedient not to take into account the flow structure and to find the upper limit of the drag.

The equations of gasdynamics for dimensionless variables are

$$\left.\begin{array}{l}\oint_L y^\nu[\rho uv\,\mathrm{d}x-(p+\rho u^2)\,\mathrm{d}y]=0, \quad \oint_L y^\nu\rho(v\,\mathrm{d}x-u\,\mathrm{d}y)=0\\ \rho\ =\psi W^{1/(\chi-1)}, \quad p=\psi W^{\chi/(\chi-1)}, \quad W=\dfrac{\chi+1}{2\chi}-\dfrac{\chi-1}{2\chi}(u^2+v^2),\\ \psi=\dfrac{\rho^{\chi k}}{p^k}, \quad k=\dfrac{1}{\chi-1}\end{array}\right\} \quad (1)$$

where

- L boundary of the arbitrary region of the flow which can be multiply connected
- x, y Cartesian co-ordinates
- u, v the corresponding velocity vector components
- ρ density
- p pressure
- ψ an entropy function
- χ an isentropic exponent
- ν either zero or one in two-dimensional or axisymmetrical cases respectively.

Let the head part of the body be restricted by the inequalities

$$0\leqq x\leqq X, \quad 0\leqq y\leqq Y,$$

where X, Y are given numbers. Choose the control contour and denote the Mach line of the uniform initial flow by $\breve{s}a$ (Fig. 1). Point a is the front point of a sharpened profile. The attached shock waves can originate at this point. If we have a detached shock wave then let point a be the point of intersection of the shock wave and a stream line that separates the gas mass flowing into the body. The remaining part of the control contour that can pass through the gas is ah. The contour $\breve{s}ah$ may be closed by the axis of symmetry and the contour of the body.

The choice of the control contour defines the role of the head shock wave in the drag increase. If the gas without shock waves acts on the body to get maximum drag then the results of the solution of the variational problem will give further conclusions about the drag limit.

Write the equation of line ah as $y=f(x)$. According to the first equation of (1) the resultant for X in the x direction is

$$X=\frac{(\chi+1)y_a^{\nu+1}}{2\chi(\nu+1)}(1+w_\infty^2)+\int_{x_a}^{x_h} f^\nu[(p+\rho u^2)f'-\rho uv]\,\mathrm{d}x \quad (2)$$

where w_∞ is the velocity of the initial flow. The last three equations of (1) are used to convert the first term of the right-hand side of equation (2). If the summary mass flow through the contour $\tilde{s}ah$ is zero the second equation of (1) gives

$$\Psi = 0 = \frac{w_\infty y_a^{\nu+1}}{\nu+1} + \int_{x_a}^{x_h} f^\nu \rho(uf' - v)\,\mathrm{d}x \tag{3}$$

To formulate the variational problem for the body maximum drag it is necessary to use equations (2) and (3), the equations of gasdynamics (1), the relations along the permissible discontinuities and the boundary conditions of the problem. Such a complete problem is not considered.

Let us consider the problem based on equations (2) and (3) only, inequalities

$$0 \leqq f(x) \leqq Y, \quad 0 \leqq x \leqq X \tag{4}$$

and the obvious inequality

$$\psi \leqq \psi_\infty \tag{5}$$

which expresses the increase or conservation of the entropy behind a shock wave.

The following variational problem is formulated. We need to find the functions $f(x)$, $u(x)$, $v(x)$, $\psi(x)$ maximising X, equation (2), in accordance with conditions (3)–(5) and the given values w_∞, X, Y. Let

$$T = X + \lambda\Psi$$

$$T = \frac{F}{\nu+1} y_a^{\nu+1} + \int_{x_a}^{x_h} \phi(f', f, w, \theta, \psi)\,\mathrm{d}x$$

$$F = \frac{(\chi+1)(1+w_\infty^2)}{2\chi} + \lambda w_\infty$$

$$\phi = f^\nu[(p+\rho w^2\cos^2\theta)f' - \rho w^2 \sin\theta\cos\theta - \lambda\rho w(\sin\theta - f'\cos\theta)]$$

where λ is constant Lagrange multiplier, w is the velocity, θ is the angle between the velocity vector and the axis x.

First suppose that $\psi = \psi_\infty$. Calculate the first variation

$$\delta T = (Fy^\nu - \phi_{f'})_a \delta y_a - \phi_a \delta x_a + (\phi_{f'})_h \delta y_h$$
$$+ \int_{x_a}^{x_h} \left[\left(\phi_f - \frac{\mathrm{d}}{\mathrm{d}x}\phi_{f'}\right)\delta f + \phi_w\,\delta w + \phi_\theta\,\delta\theta\right]\mathrm{d}x \tag{6}$$

where suffixes f, f', w, θ indicate the partial derivatives,

$$(Fy^\nu - \phi_{f'})_a = y_a^\nu[F - (p + \rho w^2\cos^2\theta + \lambda\rho w\cos\theta)_{ah}]$$

$$(\phi_{f'})_h = y_h^\nu(p + \rho w^2\cos^2\theta + \lambda\rho w\cos\theta)_h$$

Double suffix *ah* indicates values at point *a* when they tend to it from point *h*.

All the other values in equation (6) are

$$\phi_f - \frac{\mathrm{d}}{\mathrm{d}x}\phi_{f'} = \frac{v\phi}{f} - \frac{\mathrm{d}}{\mathrm{d}x}[f^v(p+\rho w^2\cos^2\theta + \lambda\rho w\cos\theta)]$$

$$\phi_w = -\frac{f^v\rho}{\chi+1-(\chi-1)w^2}\{2w(\chi+1+\chi w^2)(\sin\theta - f'\cos\theta)\cos\theta$$
$$+[\chi+1-(\chi-1)w^2]wf' + \lambda(\chi+1)(1-w^2)(\sin\theta - f'\cos\theta)\}$$

$$\phi_\theta = -f^v\rho[w^2(\cos 2\theta + f'\sin 2\theta) + \lambda w(\cos\theta + f'\sin\theta)]$$

The necessary condition of maximum X is $\delta T \leqq 0$ at the permissible variations. Equation (6) shows that this condition is satisfied when

$$x_a = 0, \quad y_h = Y \tag{7}$$

and

$$\phi_\alpha \geqq 0, \quad (\phi_{f'})_h \geqq 0 \tag{8}$$

$$(Fy^v - \phi_{f'})_a = 0 \tag{9}$$

$$\phi_f - \frac{\mathrm{d}}{\mathrm{d}x}\phi_{f'} = 0, \quad \phi_w = 0, \quad \phi_\theta = 0 \qquad (0 \leqq x \leqq X) \tag{10}$$

Inequalities (8) give $\delta T \leqq 0$ because the permissible variations satisfy the condition $\delta x_a \geqq 0$, $\delta y_h \leqq 0$, when equalities (7) are satisfied.

Functions $f(x)$, $w(x)$, $\theta(x)$ at $0 \leqq x \leqq X$ and the value of y_a are determined by equation (10) and equality (9). The value of λ is determined by equality (3). The conditions (8) must then be checked.

The system of equalities

$$(f')^{-1} = 0, \quad w = 1, \quad \theta = \pi$$

gives one partial solution in two-dimensional and axisymmetrical cases. In this case the gas is thrown away towards the initial flow with sonic velocity. As a matter of fact this flow can not be received at the finite length of the body.

In a two-dimensional case the value ϕ_f and the first equation of (10) gives

$$p + \rho w^2\cos^2\theta + \lambda\rho w\cos\theta = \text{const}$$

This equality together with the second and the third equations of (10) shows that in the two-dimensional case, values of f', w and θ are constant.

The equalities

$$f(x) = Y, \quad \theta = \arccos\left[-\left(\frac{(\chi+1)(w^2-1)}{2w^2}\right)^{1/2}\right] \tag{11}$$

give one more partial solution. This solution stipulates capture of the maximum mass flow by the diffuser and throwing the gas through the contour that is parallel to the velocity vector of the initial flow.

In the axisymmetrical case the variety of the solutions is wider.

Let the entropy increase be permissible. Denote $\psi = \psi_\infty \psi_*$ where $0 \leqq \psi_* \leqq 1$. In the equality (6) for the first variation there appears a new term

$$\delta T_* = \int_{x_a}^{x_h} \phi \frac{\delta \psi_*}{\psi_*} \mathrm{d}x$$

Substitute in this term the functions found at $\psi = \psi_\infty$. The permissible variation $\delta\psi_*$ satisfies the condition $\delta\psi_* \leqq 0$. Therefore the condition $\delta T \leqq 0$ is satisfied at

$$\phi \geqq 0 \qquad (0 \leqq x \leqq X) \tag{12}$$

After solving equations (3), (9), (10) we must check that conditions (8) and (12) are satisfied. The satisfaction of condition (12) provides the satisfaction of the first condition of (8).

Examples of calculations were computed for $\chi = 1{\cdot}4$ for two-dimensional and axisymmetrical cases. For every supersonic value w_∞ and for every value $L = X/Y$ conditions (8) and (12) are satisfied. Therefore at least at $\chi = 1{\cdot}4$ the maximum drag is provided when the gas did not pass the shock wave.

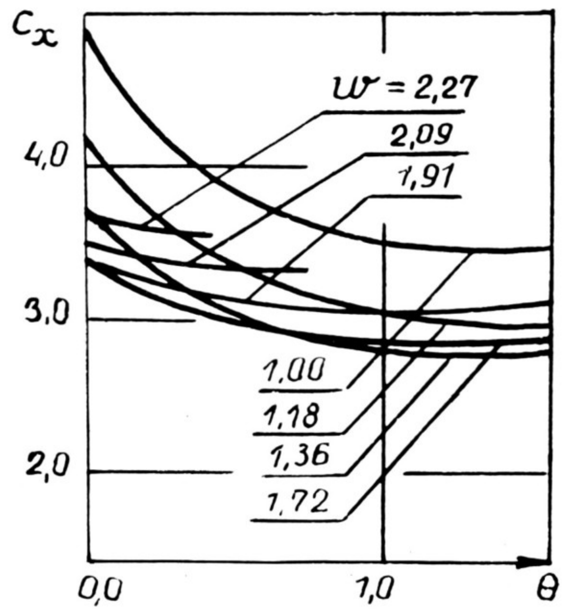

FIG. 2.

The results of the calculations are shown on Fig. 2 where the drag coefficient c_x and the parameter Θ are

$$c_x = 2X/w_\infty^2 Y, \quad \Theta = \operatorname{arc\,tan} f'$$

The drag coefficient to the accuracy of Fig. 2 for bodies of revolution does not differ from the two-dimensional case.

We note, for example, that for Mach number of the initial flow of 4, the maximum drag of a body of revolution is twice as much as the wave drag of a disc in the axisymmetrical case.